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MATHEMATICS HONS. PAPER-III

GROUP-A : REAL ANALYSIS

Lecture - 01

Contents: Real numbers, closed interval, open interval, Modulus, Upper bound, least upper bound, lower bound, greatest lower bound, Bounded set. Axiom of least upper bound, Axiom of greatest lower bound.

Definition 1. Open interval: $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

Closed interval: $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

Definition 2. If x be a real number, then its absolute value or modulus, denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Properties of Modulus:-

1. $|x| = \max\{-x, x\}$, for any real number x.

2. $|x|^2 = x^2 = |-x|^2$, for any real number x

3. $|x| = |-x|$, $\forall\, x \in \mathbb{R}$.

4. For all real numbers x & y,
$$|x+y| \leq |x| + |y|$$

5.

5. For all real numbers x & y

$$||x| - |y|| \leq |x - y|$$

Bounds for sets of real numbers

Definition: → Let S be a non-empty set of real numbers. A real number u is called an upper bound of S,
if $x \leq u, \ \forall \ x \in S$.

Remark: If a set has an upper bound then it is said to be bounded above.

Definition: → Let S be a non-empty set of real numbers. A real number l, is called a lower bound of S,
if $l \leq x, \ \forall \ x \in S$.

Remark: - If a set has a lower bound, then it is said to be bounded below.

Definition: - If S be be a non-empty set of real numbers. Then it is said to be bounded if and only if it is bounded above and bounded below

Definition: - (Least upper bound/l.u.b or supremum)

Let S be a non-empty set of real numbers. A real number M is called a least upper bound of S if

(i) M is an upper bound of set S.

(ii) For any upper bound M_1 of S, $M \leq M_1$

Definition : (Greatest lower bound (g.l.b) or infimum):-

Let S be a non-empty set of real numbers. Then a real number L is called a greatest lower bound of S if

(i) L is a lower bound of the set S.

(ii) For any lower bound L_1 of S,

$$L \geq L_1.$$

Example (1) :- Let the set $S = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$

0 is a lower bound of S.

g.l.b of $S = 0$

1 is an upper bound of S.

l.u.b of $S = S$.

Here $0 \notin S$ but $1 \in S$.

Example 2. - A finite set of real numbers is always bounded.

Example 3 :- The set $\mathbb{N} = \{1, 2, 3, \dots\}$ of Natural numbers is bounded below.

$\therefore$ g.l.b of $\mathbb{N} = 1 \in \mathbb{N}$.

$\mathbb{N}$ is not bounded above.

Exercise (1) :- Prove that for a non-empty set of real numbers

(i) if a least upper bound exists then it is unique.

(ii) if a greatest lower bound exists then it is unique.

Axiom I : (Axiom of least upper bound) Any non-empty set of real numbers which is bounded above has a least upper bound (l.u.b.).

Axiom II : (Axiom of greatest lower bound) Any non-empty set of real numbers which is bounded below has a greatest lower bound.

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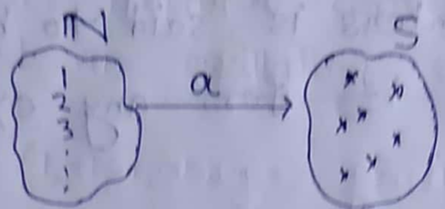
MATHEMATICS HONS. PAPER-III

GROUP-A : REAL ANALYSIS

Lecture -02

Contents: Sequence, Real Sequence, Bounded sequence, convergent sequence, divergent Sequence,

Definition:- A sequence in a non-empty set S is a mapping $a: \mathbb{N} \to S$, where $\mathbb{N}$ is the set of all real numbers.



The image of an element n of $\mathbb{N}$ under the mapping a is denoted by $a(n)$ or a_n is called the nth term of the sequence.

Such a sequence is usually denoted by (a_n) or $\{a_1, a_2, \ldots\ldots, a_n, \ldots\ldots\}$ or $\{a_n\}_{n=1}^{\infty}$.

Definition:→ If $S = \mathbb{R}$ (set of all real numbers) then the sequence (a_n) is called a real sequence or sequence of real numbers.

Example① $\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ i.e; $(a_n) = (\frac{1}{n})$

② $\{\frac{1}{2}, \frac{2}{3}, \cdots, \frac{n}{n+1}, \cdots\}$ i.e; $(a_n) = (\frac{n}{n+1})$

Definition:- A sequence (a_n) is said to be bounded if its range $\{a_n : n \in \mathbb{N}\}$ is a bounded set.

If a sequence (a_n) is not bounded then it is called an unbounded set.

Example:- The sequence $(a_n) = (\frac{1}{n})$ is bounded sequence.

Definition (Convergent sequence):→ A sequence a_n of real numbers is said to be convergent to the limit l if for any $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$ such that

$$|a_n - l| < \epsilon, \ \forall n \geq N$$

Example ① $\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}\}$ i.e; $(\frac{1}{n})$ is convergent to the limit 0

② $\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{n}{n+1}, \cdots\}$ i.e; $(a_n) = (\frac{n}{n+1})$ is convergent to the limit 1.

Divergent sequence :-

(i) A sequence (a_n) is said to be divergent to $+\infty$ if for any M (large number), $\exists N$ s.t.

$$a_n > M, \ \forall n \geq N$$

(ii) A sequence (a_n) of real number is said to be divergent to $-\infty$ if for any M (large number), $\exists N$ s.t.

$$a_n < -M, \ \forall \ n \geq N$$

(iii) A sequence (a_n) is said to be divergent if it diverges to $+\infty$ or to $-\infty$.

Example ① (n^2) is diverges to $+\infty$

(ii) $(-n^2)$ is diverges to $-\infty$

(iii) (n^2) and $(-n^2)$ are divergent.

Definition (Oscillating Sequence) :- A sequence which is neither convergent nor divergent is called an oscillating sequence.

Example ① $\{1, -1, 1, -1, \cdots\}$ is oscillating

② $\{1, -2, 3, -4, \cdots\}$ is oscillating

THEOREM ①: Proved that every convergent sequence of real numbers is a bounded sequence.

Proof:→ Let (a_n) be a convergent sequence of real numbers converging to the limit l.

$\therefore$ For any $\epsilon > 0$, $\exists N(\epsilon)$ such that

$$|a_n - l| < \epsilon, \ \forall \ n \geq N \in \mathbb{N}.$$

$$\therefore |a_n| = |a_n - l + l|$$

$$\leq |a_n - l| + |l|$$

$\therefore |a_n| < \epsilon + |l|, \forall n \geq N$

Let $M = \max\{|a_1|, |a_2|, \cdots\cdots, |a_{N-1}|\}$

If $A = \max\{M, \epsilon + |l|\}$, then

$|a_n| < A, \forall n \in \mathbb{N}$

$\therefore$ The sequence (a_n) is <u>bounded</u>.

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<u>Remark</u>:

<u>converse not true (example)</u>:-

Let $(a_n) = \{1, -1, 1, -1, \cdots\}$

This sequence is bounded as

$-2 < a_n < 2, \quad \forall n \in \mathbb{N}$.

But it is <u>not convergent</u>.

It is an oscillating sequence.

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B.Sc. - II (MATHEMATICS HONS)
Paper - III
Group A : Real Analysis - I

Contents:→ Uniqueness of limit of a convergent sequence, Monotonic increasing sequence, Monotonic decreasing sequence etc.

THEOREM:→ Proved that the limit of a convergent sequence is unique.

Proof:→

Let l and l' be two limits of the convergent sequence (a_n).

$\therefore$ For any $\epsilon > 0$, $\exists$ a natural number $N_1(\epsilon)$ and $N_2(\epsilon)$ s.t.

$$|a_n - l| < \frac{\epsilon}{2}, \ \forall\ n \geq N_1 \ \&$$

$$|a_n - l'| < \frac{\epsilon}{2}, \ \forall\ n \geq N_2$$

Let $N = \max\{N_1, N_2\}$

$$\therefore |l - l'| = |l - a_n + a_n - l'|$$

$$\leq |a_n - l| + |a_n - l'|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \ \forall\ n \geq N$$

$\therefore |l - l'| < \epsilon, \ \forall\ n \geq N$

$\because \epsilon$ is arbitrary

$\therefore |l - l'| = 0 \Rightarrow l = l'$ proved.

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Monotonic increasing sequence :→ A sequence (a_n) of real numbers is said to be a monotonic increasing sequence if

$$a_n \leq a_{n+1},\ \forall n$$

Example :→ (i) $(1, 2, 3, \cdots, n, \cdots)$

(ii) $(1, 1, 2, 2, 3, 3, 4, 4, \cdots)$

If $a_n < a_{n+1}, \forall n$, then (a_n) is called strictly monotonic increasing sequence.

Monotonic decreasing sequence :→ A sequence (a_n) of real numbers is said to be a monotonic decreasing sequence if

$$a_{n+1} \leq a_n,\ \forall n$$

If $a_{n+1} < a_n, \forall n$, then (a_n) is called strictly monotonic decreasing sequence.

THEOREM :→ Proved that a monotonic increasing sequence bounded above is convergent (to l.u.b. of the set of points of the sequence)

Proof :→ Let (a_n) be a real sequence which is monotonic increasing and bounded above.

$\because$ (a_n) is bounded above,

$\therefore$ $\exists$ a real number K such that

$$a_n \leq K, \forall n$$

$\therefore$ The set S of points of the sequence (a_n) is also bounded above and non-empty.

Hence by l.u.b axiom (If a non-empty set of real numbers is bounded above then it has a l.u.b)

$\therefore$ S has a l.u.b.

Let $l =$ l.u.b of S

$\therefore$ ① $a_n \leq l, \forall n$

$\Rightarrow a_n < l + \epsilon, \forall n$, for any $\epsilon > 0$

② There is at least a_N s.t.

$$a_N > l - \epsilon$$

But since (a_n) is monotonic increasing sequence.

$$\therefore a_n \geq a_N, \forall n \geq N$$

$$\therefore a_n > l - \epsilon, \forall n \geq N$$

From ① & ② implies that

$$l - \epsilon < a_n < l + \epsilon, \forall n \geq N$$

$$\Rightarrow |a_n - l| < \epsilon, \forall n \geq N$$

$\therefore$ (a_n) is convergent to the limit l.

\# proved.

THEOREM:→ Proved that a monotonic decreasing Sequence which is bounded below is convergent to the g.l.b. of the set of points of the sequence.

Proof:→ Let (a_n) be a monotonic decreasing sequence which is bounded below.

$\therefore \exists$ a real number K s.t. $K \leq a_n, \forall n$.

If S be the set of points of the sequence, then S is non-empty and bounded below by K.

$\therefore$ By g.l.b axiom of real numbers, S has a g.l.b. (If S is non-empty set of real numbers which is bounded below then it has a g.l.b)

Let l = g.l.b of S

$\therefore$ ① $l \leq a_n, \forall n$

$\Rightarrow l - \epsilon < a_n, \forall n$, for any $\epsilon > 0$,

② For any $\epsilon > 0$, $\exists$ at least one a_N s.t.

$a_N < l + \epsilon$

$\because (a_n)$ is monotonic decreasing

$\therefore a_n \leq a_N$ if $n \geq N$

$\therefore a_n < l + \epsilon, \forall n \geq N$

From ① & ②, we have

$l - \epsilon < a_n < l + \epsilon, \forall n \geq N$

$\therefore |a_n - l| < \epsilon, \forall n \geq N$

$\therefore (a_n)$ is convergent to l. proved.

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B.Sc-II
MATHEMATCS HONS : Paper-III
Group A : Real Analysis

Contents :→ Theorems related to monotonic sequence; Cauchy sequence. Cauchy general Principle for convergence.

THEOREM :→ A monotonic ~~sequ~~ increasing sequence which is not bounded above is divergent (to $+\infty$)

Proof :→ Let (a_n) be a monotonic increasing sequence which is not bounded above.

$\therefore$ For any M (however large) $\exists$ a_N s.t.

$$a_N > M$$

But since (a_n) is monotonic increasing

$$\therefore \; a_n \geq a_N, \; \forall \; n \geq N$$

$$\therefore \; a_n > M, \; \forall \; n \geq N$$

$\therefore$ (a_n) is divergent to $+\infty$. proved. #

THEOREM :→ A monotonic decreasing sequence which is not bounded below is divergent (to $-\infty$).

Proof: → Let (a_n) be monotonic decreasing sequence which is not bounded below.

∴ For any M (however small), there exists a_N such that $a_N < M$.

But since, (a_n) is monotonic decreasing

$$\therefore a_n \leq a_N, \ \forall \ n \geq N$$

$$\therefore a_n < M, \ \forall \ n \geq N$$

∴ (a_n) is divergent to $-\infty$.

proved.

THEOREM : → A monotonic sequence (a_n) is convergent if and only if (a_n) is bounded.

Proof: → (⇒)

(i) Every convergent sequence is bounded.

(⇐) Conversely,

(ii) m.i. sequence bounded above is convergent.

(iii) m.d. sequence bounded below is convergent.

Cauchy sequence : → A sequence (a_n) of real numbers is said to be a cauchy sequence if for any $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$ s.t.

$$|a_n - a_m| < \epsilon, \ \forall \ m, n \geq N.$$

THEOREM: (Necessary and sufficient condition)

A sequence (a_n) of real numbers is convergent if and only if it is a Cauchy sequence.

Proof: $\rightarrow (\Rightarrow)$

Let (a_n) be a convergent sequence converging to limit l.

$\therefore$ For any $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$ s.t.

$$|a_n - l| < \frac{\epsilon}{2}, \ \forall n \geq N$$

$$\& \ |a_m - l| < \frac{\epsilon}{2}, \ \forall n \geq N$$

$$\therefore |a_n - a_m| = |a_n - l + l - a_m|$$

$$\leq |a_n - l| + |a_m - l|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \ \forall m, n \geq N$$

$$\therefore |a_n - a_m| < \epsilon, \ \forall m, n \geq N$$

$\therefore (a_n)$ is a Cauchy sequence.

($\Leftarrow$) Conversely

Let (a_n) be a Cauchy sequence.

$\therefore$ For any $\epsilon > 0$, $\exists N(\epsilon)$ s.t.

$$|a_n - a_m| < \frac{\epsilon}{2}, \ \forall m, n \geq N$$

$$\therefore |a_n - a_N| < \frac{\epsilon}{2}, \ \forall n \geq N$$

$$\therefore a_N - \frac{\epsilon}{2} < a_n < a_n + \frac{\epsilon}{2}, \ \forall n \geq N$$

Let $S = \{x \in \mathbb{R} \mid x < a_n$ for an infinite number of members of $(a_n)\}$

Now,

$$\left(a_N - \frac{\epsilon}{2}\right) \in S, \quad \therefore S \neq \phi.$$

Also $\left(a_N + \frac{\epsilon}{2}\right)$ is an upper bound of S.

By the axiom of l.u.b of real numbers (A non-empty set which is bounded above has a l.u.b.)

$\therefore$ S has a l.u.b.

Let l.u.b of $S = l$

$$\therefore l \in \left[a_N - \frac{\epsilon}{2}, a_N + \frac{\epsilon}{2}\right]$$

$$\therefore |a_N - l| < \frac{\epsilon}{2}$$

$$\therefore |a_n - l| = |a_n - a_N + a_N - l|$$

$$\leq |a_n - a_N| + |a_N - l|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall n \geq N$$

$$\therefore |a_n - l| < \epsilon, \quad \forall n \geq N$$

$\therefore (a_n)$ is convergent to l.

proved.

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